5.1: Fundamental Identities

I. Identities

A. An identity is a true sentence in which *x* must be in the domain of both expressions.

B. Examples:

$$\frac{x^3-1}{x-1}=x^2+x+1 \qquad \frac{2x^3}{x^2}=2x$$

$$\csc x=\frac{1}{\sin x} \qquad \cot x=\frac{\cos x}{\sin x}$$

C. The set of all values for which the identity is true is called the *domain of validity*.

II. Basic Trigonometric Identities

A. Reciprocal Identities

$$\csc\theta=\frac{1}{\sin\theta} \quad \sec\theta=\frac{1}{\cos\theta} \quad \cot\theta=\frac{1}{\tan\theta}$$

$$\sin\theta=\frac{1}{\csc\theta} \quad \cos\theta=\frac{1}{\sec\theta} \quad \tan\theta=\frac{1}{\cot\theta}$$

B. Quotient Identities

$$\tan\theta=\frac{\sin\theta}{\cos\theta} \quad \cot\theta=\frac{\cos\theta}{\sin\theta}$$

III. Pythagorean Identities

A. Since the sine and cosine of an angle make up the legs of a reference triangle with a hypotenuse of 1:

$$\boxed{\cos^2 t+\sin^2 t=1}$$

B. If we divide through by the cosine squared:

$$\frac{\cos^2 t}{\cos^2 t}+\frac{\sin^2 t}{\cos^2 t}=\frac{1}{\cos^2 t}\Rightarrow \boxed{1+\tan^2 t=\sec^2 t}$$

C. If we divide through by the sine squared:

$$\frac{\cos^2 t}{\sin^2 t}+\frac{\sin^2 t}{\sin^2 t}=\frac{1}{\sin^2 t}\Rightarrow \boxed{\cot^2 t+1=\csc^2 t}$$

C. Pythagorean Identities

$$\boxed{\begin{gathered}\cos^2 x+\sin^2 x=1\\ 1+\tan^2 x=\sec^2 x\\ \cot^2 x+1=\csc^2 x\end{gathered}}$$

D. Example 1: use the identities to find *cos(x)* and *tan(x)* if *sin(x)=1/3* and *cos(x)<0*.

note:we could use a reference triangle, but are using identities instead.

$$\cos^2 x+\sin^2 x=1$$

$$\cos^2 x+\left(\frac{1}{3}\right)^2=1$$

$$\cos^2 x+\frac{1}{9}=1$$

$$\cos^2 x=\frac{8}{9}$$

$$\cos x=\pm\frac{2\sqrt{2}}{3}$$

$$\boxed{\cos x=-\frac{2\sqrt{2}}{3}}$$

$$\cos x = \frac{2\sqrt{2}}{3}$$
$$\sec x = \frac{3}{2\sqrt{2}}$$
$$1+\tan^2 x = \sec^2 x$$
$$1+\tan^2 x = \left(-\frac{3}{2\sqrt{2}}\right)^2$$
$$1+\tan^2 x = \frac{9}{8}$$
$$\tan^2 x = \frac{1}{8}$$
$$\tan x = \pm\frac{\sqrt{2}}{4}$$
$$\boxed{\tan x = -\frac{\sqrt{2}}{4}}$$

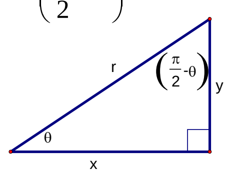
E. Example 2: Write the following expression in terms of *sin t:*

$$\sin^4 t + \cos^4 t$$
$$\sin^4 t + \left(\cos^2 t\right)^2$$
$$\sin^4 t + \left(1-\sin^2 t\right)^2$$
$$\sin^4 t + 1 - 2\sin^2 t + \sin^4 t$$
$$\boxed{2\sin^4 t - 2\sin^2 t + 1}$$

IV. Cofunction Identities

Note that in the triangle below,

$$\sin\theta = \frac{y}{r} = \cos\left(\frac{\pi}{2}-\theta\right) \text{ and } \cos\theta = \frac{x}{r} = \sin\left(\frac{\pi}{2}-\theta\right)$$



This is true for all co-functions , and can be generalized as follows:

The Cofunction Identities

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \quad \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$
$$\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta \quad \sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$$
$$\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta \quad \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$$

V. Odd-Even Identities

$$\sin(-\theta) = -\sin\theta \quad \cos(-\theta) = \cos\theta$$
$$\csc(-\theta) = -\csc\theta \quad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \quad \cot(-\theta) = -\cot\theta$$

VI. Simplifying Trigonometric Expressions

A. Example 4: Simplify each of the following:

1. $(\sec t - \tan t)(\sec t + \tan t) = \sec^2 t - \tan^2 t$

$$= \left(1+\tan^2 t\right) - \tan^2 t = \boxed{1}$$

2. $\dfrac{\sin^2 t - 2\sin t + 1}{1-\sin t} = \dfrac{(\sin t - 1)^2}{-(\sin t - 1)}$

$$= \frac{\sin t - 1}{-1} = \boxed{1-\sin t}$$

$$3.\ \left(\frac{\sin t}{\cos t}\right)^2 - \frac{1}{\cos^2 t} = \frac{\sin^2 t}{\cos^2 t} - \frac{1}{\cos^2 t}$$
$$= \frac{\sin^2 t - 1}{\cos^2 t}$$
$$= \frac{-\left(1-\sin^2 t\right)}{\cos^2 t}$$
$$= \frac{-\cos^2 t}{\cos^2 t} = \boxed{-1}$$

VII. Solving Trigonometric Equations

A. Example 5: Solve the following trigonometric equations over the interval [0,2pi)

$$1.\ (\sec x)(\csc x) = 2\csc x$$
$$(\sec x)(\csc x) - 2\csc x = 0$$
$$\csc x(\sec x - 2) = 0$$

$\csc x = 0$ no solution

$\sec x - 2 = 0$

$\sec x = 2$

$$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$$

$$2.\ 1 - 3\tan^2 x = 0$$
$$\tan^2 x = \frac{1}{3}$$
$$\tan x = \pm\frac{1}{\sqrt{3}}$$

$\tan x = \frac{1}{\sqrt{3}}$ $\quad x = \frac{\pi}{6}, \frac{7\pi}{6}$

$\tan x = -\frac{1}{\sqrt{3}}$ $\quad x = \frac{5\pi}{6}, \frac{11\pi}{6}$

$$x = \boxed{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$$

$$3.\ \sin^2 3x = 0$$
$$\sin 3x = 0$$

$$3x = 0 + 2n\pi$$
$$3x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots$$
$$x = \ldots, -\frac{2\pi}{3}, \frac{0}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \ldots$$

$$3x = \pi + 2n\pi$$
$$3x = \ldots, -\pi, \pi, 3\pi, 5\pi, 7\pi, \ldots$$
$$x = \ldots, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$\boxed{x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}}$$

B. Find all solutions to the following trigonometric identities

$$1.\ \csc^2 x + \csc x - 2 = 0$$
$$(\csc x - 1)(\csc x + 2) = 0$$

$\csc x = 1$ $\quad x = \frac{\pi}{2}$

$\csc x = -2$ $\quad x = \frac{7\pi}{6}, \frac{11\pi}{6}$

to include all solutions, we write:

$$\boxed{x = \frac{\pi}{2} + 2n\pi, \frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi}$$

$$2.\ \cos^2 x + \sin x + 1 = 0$$
$$1 - \sin^2 x + \sin x + 1 = 0$$
$$\sin^2 x - \sin x - 2 = 0$$
$$(\sin x + 1)(\sin x - 2) = 0$$

$\sin x = -1$ $\quad x = \frac{3\pi}{2}$

$\sin x = 2$ no solution

to include all solutions, we write:

$$\boxed{x = \frac{3\pi}{2} + 2n\pi}$$

5.2: Proving Trigonometric Identities

I. Proving Identities

A. Steps:

1. Begin with the expression on one side of the identity (usually start with the side that looks more complicated)
2. Show a sequence of expressions, each of which is clearly equivalent to the one before it.* If you don't know what to do, convert the entire expression into sines and cosines
3. End with the expression on the other side of the identity.

*you can only change one side of the identity in a given proof

B. Prove the following identities:

1. $\sec x = \frac{\csc(-x)}{\cot(-x)}$

$$\boxed{\frac{\csc(-x)}{\cot(-x)}} = \frac{-\csc(x)}{-\cot(x)}$$
$$= \frac{\csc(x)}{\cot(x)}$$
$$= \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}$$
$$= \frac{1}{\sin x} \div \frac{\cos x}{\sin x}$$
$$= \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x}$$
$$= \frac{1}{\cos x}$$
$$= \boxed{\sec x}$$

2. $\frac{\sin x}{\csc x} = 1 - \frac{\cos x}{\sec x}$

$$\boxed{1 - \frac{\cos x}{\sec x}} = 1 - \frac{\cos x}{\frac{1}{\cos x}}$$
$$= 1 - \cos x \div \frac{1}{\cos x}$$
$$= 1 - \cos^2 x$$
$$= \sin^2 x$$
$$= \sin x \cdot \sin x$$
$$= \sin x \cdot \frac{1}{\csc x}$$
$$= \boxed{\frac{\sin x}{\csc x}}$$

3. $\sin^3 x \cos x - \sin^5 x \cos x = \sin^3 x \cos^3 x$

$$\boxed{\sin^3 x \cos x - \sin^5 x \cos x} = \sin^3 x \cos x \left(1 - \sin^2 x\right)$$
$$= \sin^3 x \cos x \left(\cos^2 x\right)$$
$$= \boxed{\sin^3 x \cos^3 x}$$

4. $\frac{1 - \tan^2 x}{1 - \sec^2 x} = 1 - \cot^2 x$

$$\boxed{\frac{1 - \tan^2 x}{1 - \sec^2 x}} = \frac{1 - \tan^2 x}{-\left(\sec^2 x - 1\right)}$$
$$= \frac{1 - \tan^2 x}{-\tan^2 x}$$
$$= -\frac{1}{\tan^2 x} + \frac{-\tan^2 x}{-\tan^2 x}$$
$$= -\cot^2 x + 1$$
$$= \boxed{1 - \cot^2 x}$$

5. $-4\sec x\tan x = \frac{1-\sin x}{1+\sin x} - \frac{1+\sin x}{1-\sin x}$

$$\boxed{\frac{1-\sin x}{1+\sin x} - \frac{1+\sin x}{1-\sin x}} = \frac{1-\sin x}{1+\sin x}\left(\frac{1-\sin x}{1-\sin x}\right) - \frac{1+\sin x}{1-\sin x}\left(\frac{1+\sin x}{1+\sin x}\right)$$

$$= \frac{1-2\sin x+\sin^2 x}{1-\sin^2 x} - \frac{1+2\sin x+\sin^2 x}{1-\sin^2 x}$$

$$= \frac{1-2\sin x+\sin^2 x-\left(1+2\sin x+\sin^2 x\right)}{1-\sin^2 x}$$

$$= \frac{1-2\sin x+\sin^2 x-1-2\sin x-\sin^2 x}{\cos^2 x}$$

$$= \frac{-4\sin x}{\cos^2 x} = -4\frac{1}{\cos x}\frac{\sin x}{\cos x} = \boxed{-4\sec x\tan x}$$

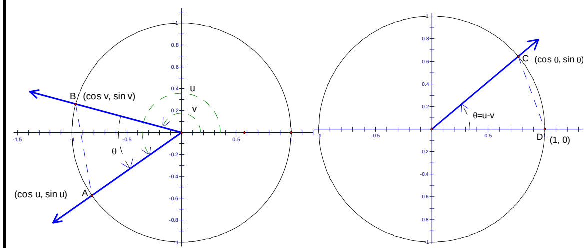
C. Disproving non-identities:

1. You can check if identities work by graphing (see exploration 1 on page 458).
2. To disprove identities, you need only find one counterexample.

5.3: Sum and Difference Identities

I. Cosine of a Difference:

A. Derivation: Consider the two diagrams, (Note that for the chords AB=CD).



$$AB = CD$$

$$\sqrt{(\cos v - \cos u)^2 + (\sin v - \sin u)^2} = \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}$$

$$\cos^2 u - 2\cos u\cos v + \cos^2 v + \sin^2 u - 2\sin u\sin v + \sin^2 v$$
$$= \cos^2\theta - 2\cos\theta + 1 + \sin^2\theta$$

$$\left(\cos^2 u + \sin^2 u\right) + \left(\cos^2 v + \sin^2 v\right) - 2\cos u\cos v - 2\sin u\sin v$$
$$= \left(\cos^2\theta + \sin^2\theta\right) + 1 - 2\cos\theta$$

$$2 - 2\cos u\cos v - 2\sin u\sin v = 2 - 2\cos\theta$$

$$\cos u\cos v + \sin u\sin v = \cos\theta$$

$$\boxed{\cos u\cos v + \sin u\sin v = \cos(u - v)}$$

B. Find the formula for the cosine of a sum.

$$\cos(u+v) = \cos[u - (-v)]$$
$$= \cos u\cos(-v) + \sin u\sin(-v)$$
$$= \cos u\cos v + \sin u(-\sin v)$$
$$= \boxed{\cos u\cos v - \sin u\sin v}$$

C. Cosine sum and difference formulas:

$$\boxed{\cos(u \pm v) = \cos u\cos v \mp \sin u\sin v}$$

D. Example 1: Determine the value of: $\cos\frac{5\pi}{12}$

First, note that $\frac{5\pi}{12} = \frac{3\pi}{4} - \frac{\pi}{3}$

$$\cos\frac{5\pi}{12} = \cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) = \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$
$$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$
$$= \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

E. Example 2: Verify the following identities:

1. $\cos(\pi + \theta) = -\cos\theta$

$$\boxed{\cos(\pi+\theta)} = \cos\pi\cos\theta - \sin\pi\sin\theta$$
$$= (-1)\cos\theta - (0)\sin\theta$$
$$= \boxed{-\cos\theta}$$

$$2.\ \cos\left(\theta-\frac{3\pi}{2}\right)=-\sin\theta$$

$$\boxed{\cos\left(\theta-\frac{3\pi}{2}\right)}=\cos\theta\cos\left(\frac{3\pi}{2}\right)+\sin\theta\sin\left(\frac{3\pi}{2}\right)$$
$$=\cos\theta(0)+\sin\theta(-1)$$
$$=\boxed{-\sin}$$

Note: these are called reduction formulas, since they reduce the complexity of the expression

II. Sine of a difference or sum

A. $\sin(u+v)=\cos\left[\frac{\pi}{2}-(u+v)\right]$

$$=\cos\left[\left(\frac{\pi}{2}-u\right)-v\right]$$
$$=\cos\left(\frac{\pi}{2}-u\right)\cos v+\sin\left(\frac{\pi}{2}-u\right)\sin v$$
$$=\sin u\cos v+\cos u\sin v$$

B. $\sin(u-v)=\cos\left[\frac{\pi}{2}-(u-v)\right]$

$$=\cos\left[\left(\frac{\pi}{2}-u\right)+v\right]$$
$$=\cos\left(\frac{\pi}{2}-u\right)\cos v-\sin\left(\frac{\pi}{2}-u\right)\sin v$$
$$=\sin u\cos v-\cos u\sin v$$

$$\boxed{\sin(u\pm v)=\sin u\cos v\pm\cos u\sin v}$$

C. Example 3: Find the value of each of the following.

1. $\sin(27.35^\circ)\cos(2.65^\circ)+\cos(27.35^\circ)\sin(2.65^\circ)$

$$=\sin(27.35^\circ+2.65^\circ)$$
$$=\sin(30^\circ)=\boxed{\frac{1}{2}}$$

2. $\cos\left(\frac{4\pi}{11}\right)\cos\left(\frac{7\pi}{11}\right)-\sin\left(\frac{4\pi}{11}\right)\sin\left(\frac{7\pi}{11}\right)$

$$=\cos\left(\frac{4\pi}{11}+\frac{7\pi}{11}\right)$$
$$=\cos(\pi)=\boxed{-1}$$

III. Tangent of a difference or sum

A. $$\boxed{\tan(u\pm v)=\frac{\sin(u\pm v)}{\cos(u\pm v)}\frac{\sin u\cos v\pm\cos u\sin v}{\cos u\cos v\mp\sin u\sin v}}$$

or

$$\boxed{\tan(u\pm v)=\frac{\tan u\pm\tan v}{1\mp\tan u\tan v}}$$

B. Prove the reduction formula below:

$$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$$

$$\boxed{\tan\left(\frac{\pi}{2}+\theta\right)}=\frac{\sin\left(\frac{\pi}{2}+\theta\right)}{\cos\left(\frac{\pi}{2}+\theta\right)}=\frac{\sin\left(\frac{\pi}{2}\right)\cos\theta+\cos\left(\frac{\pi}{2}\right)\sin\theta}{\cos\left(\frac{\pi}{2}\right)\cos\theta-\sin\left(\frac{\pi}{2}\right)\sin\theta}$$
$$=\frac{(1)\cos\theta+(0)\sin\theta}{(0)\cos\theta-(1)\sin\theta}$$
$$=\frac{\cos\theta}{-\sin\theta}$$
$$=\boxed{-\cot\theta}$$

IV. Verifying a Sinusoid Algebraically

Express the function below as a single sinusoid

$$f(x) = 2\sin 2x + 3\cos 2x$$

$$a\sin(bx+c) = a\left(\sin bx\cos c + \cos bx\sin c\right)$$
$$= a\sin bx\cos c + a\cos bx\sin c$$

$$2\sin 2x + 3\cos 2x = \left(a\cos c\right)\sin bx + \left(a\sin c\right)\cos bx$$

$$\boxed{b=2}$$

$$2 = a\cos c \qquad 3 = a\sin c$$

$$\left(a\cos c\right)^2 + \left(a\sin c\right)^2 = \left(2\right)^2 + \left(3\right)^2$$
$$a^2\cos^2 c + a^2\sin^2 c = 13$$
$$a^2\left(\cos^2 c + \sin^2 c\right) = 13$$
$$a^2 = 13$$
$$a = \pm\sqrt{13}$$
$$\text{let's use } a = \sqrt{13}$$

$$\sqrt{13}\cos c = 2 \qquad \sqrt{13}\sin c = 3$$
$$\cos c = \frac{2}{\sqrt{13}} \qquad \sin c = \frac{3}{\sqrt{13}}$$
$$c = \cos^{-1}\left(\frac{2\sqrt{13}}{13}\right) \text{ or } c = \sin^{-1}\left(\frac{3\sqrt{13}}{13}\right)$$

Exact answer:

$$\boxed{f(x) = \sqrt{13}\sin 2\left[x + \cos^{-1}\left(\frac{2\sqrt{13}}{13}\right)\right]}$$

or

$$\boxed{f(x) = \sqrt{13}\sin 2\left[x + \sin^{-1}\left(\frac{3\sqrt{13}}{13}\right)\right]}$$

Approximate:

$$\boxed{3.606\sin 2\left[x + 0.983\right]}$$

5.4: Multiple-Angle Identities

I. Double-Angle Identities

A.

$$\sin 2u = 2\sin u\cos u$$

$$\cos 2u = \begin{cases} \cos^2 u - \sin^2 u \\ 2\cos^2 u - 1 \\ 1 - 2\sin^2 u \end{cases}$$

$$\tan 2u = \frac{2\tan u}{1-\tan^2 u}$$

B. Example 1: Prove the identity below:

$$\cos 2u = \cos^2 u - \sin^2 u$$

$$\cos 2u = \cos(u+u)$$

$$= \cos u\cos u - \sin u\sin u$$

$$= \cos^2 u - \sin^2 u$$

C. Example 2: Prove the identity below.

$$\sec 2x = \frac{\sec^2 x + \sec^4 x}{2+\sec^2 x - \sec^4 x}$$

$$= \frac{\sec^2 x(1+\sec^2 x)}{(2-\sec^2 x)(1+\sec^2 x)}$$

$$= \frac{\sec^2 x}{2-\sec^2 x}$$

$$= \frac{\cos^2 x(\sec^2 x)}{\cos^2 x(2-\sec^2 x)}$$

$$= \frac{1}{2\cos^2 x - 1} \quad = \frac{1}{\cos 2x} \quad = \sec 2x$$

II. Power-Reducing Identities

A.

$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

$$\tan^2 u = \frac{1-\cos 2u}{1+\cos 2u}$$

B. Example 3: rewrite the expression below in terms of trigonometric functions with no power greater than 1.

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2$$

$$= \frac{1-2\cos 2x + \cos^2 2x}{4}$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

III. Half-Angle Identities

A.

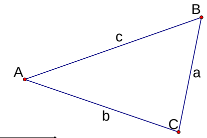
$$\sin\frac{u}{2}=\pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2}=\pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2}=\begin{cases}\pm\sqrt{\dfrac{1-\cos u}{2}}\\ \dfrac{1-\cos u}{\sin u}\\ \dfrac{\sin u}{1+\cos u}\end{cases}$$

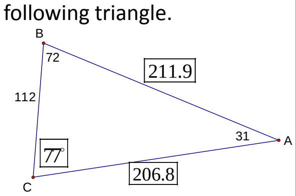
5.5: The Law of Sines

I. The Law of Sines



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

B. Example 1: Solve the following triangle.



$$m\angle C = 180 - 31 - 72 = 77$$

$$\frac{\sin 31}{112} = \frac{\sin 72}{b} = \frac{\sin 77}{c}$$

$$\frac{\sin 31}{112} = \frac{\sin 72}{b}$$

$$b \sin 31 = 112 \sin 72$$

$$b = \frac{112 \sin 72}{\sin 31}$$

$$b \approx 206.8$$

$$\frac{\sin 31}{112} = \frac{\sin 77}{c}$$

$$c = \frac{112 \sin 77}{\sin 31}$$

$$c \approx 211.9$$

C. SSA: The ambiguous case

1. Given two sides and an angle, the law of sines can give 0, 1, or 2 solutions
2. Example 2: Determine all triangles for which *a=50*, *b=150*, and *A=60* degrees.

$$\frac{\sin 60}{50} = \frac{\sin B}{150}$$

$$\frac{150 \sin 60}{50} = \sin B$$

$$3\left(\frac{\sqrt{3}}{2}\right) = \sin B$$

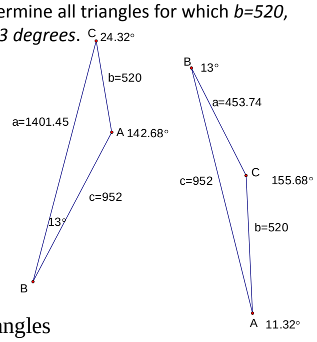
$$\frac{3\sqrt{3}}{2} = \sin B$$

$$B = \sin^{-1}\left(\frac{3\sqrt{3}}{2}\right) = \varnothing$$

$$\text{since } -1 \le \sin^{-1} x \le 1$$

no solution

3. Example 3: Determine all triangles for which *b=520*, *c=952*, and *B=13 degrees*.



$$\frac{\sin 13}{520} = \frac{\sin C}{952}$$

$$\frac{952 \sin 13}{520} = \sin C$$

$$0.41183 \approx \sin C$$

$$24.32 \text{ or } 155.68 = C$$

$\therefore$ we have 2 possible triangles

4. Example 5: Determine all triangles for which *a=10*, *c=20*, and *A=30 degrees*.

$$\frac{\sin 30}{10} = \frac{\sin C}{20}$$

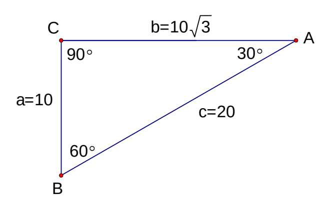
$$\sin C = \frac{20 \sin 30}{10}$$

$$\sin C = 2\left(\frac{1}{2}\right)$$

$$\sin C = 1$$

$$C = 90^\circ$$

we have the following right triangle



II. Applications

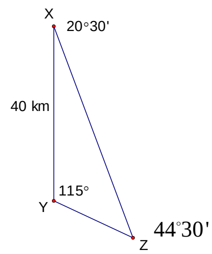
Two lighthouses at points *X* and *Y* are 40 kilometers apart. Each has a visual contact with a fishing boat at point *Z*. If the angles are as given below, how far is the fishing boat from lighthouse *Y*?

$m\angle ZXY = 20^\circ 30', m\angle ZYX = 115^\circ$

$m\angle Z = 180 - 20^\circ 30' - 115^\circ = 44^\circ 30'$

$$\frac{\sin 44^\circ 30'}{40} = \frac{\sin 115}{y}$$

$$y = \frac{40 \sin 115}{\sin 44^\circ 30'} \approx \boxed{51.72 km}$$



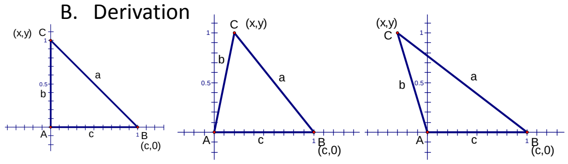
5.6: The Law of Cosines

I. Deriving the Law of Cosines

A. Let triangle ABC be any triangle with sides and angles labeled in the usual way. Then:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

B. Derivation



In each of the three cases above,

$$\frac{x}{b} = \cos a \qquad \frac{y}{b} = \sin a$$

$$x = b\cos a \qquad y = b\sin a$$

Using the distance formula:

$$a = \sqrt{(x-c)^2 + (y-0)^2}$$
$$a^2 = (x-c)^2 + y^2$$
$$a^2 = (b\cos a - c)^2 + (b\sin a)^2$$
$$a^2 = b^2\cos^2 a - 2bc\cos A + c^2 + b^2\sin^2 a$$
$$a^2 = b^2\left(\cos^2 a + \sin^2 a\right) - 2bc\cos A + c^2$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

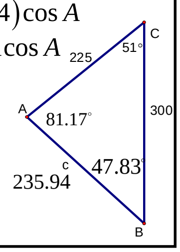
Since we chose the sides generally,
it is true for any set of sides and angles

C. Example 1: Solve the triangle with the following sides and angles. $a = 300, b = 225, C = 51^\circ$

$$c^2 = 300^2 + 225^2 - 2(300)(225)\cos 51$$
$$c^2 \approx 55666.75 \qquad \boxed{c \approx 235.94}$$

$$300^2 \approx 225^2 + 235.94^2 - 2(225)(235.94)\cos A$$
$$90000 \approx 50625 + 55666.75 - 106172.11\cos A$$
$$-16291.75 \approx -106172.11\cos A$$
$$0.153 \approx \cos A$$
$$\boxed{A \approx 81.17^\circ}$$

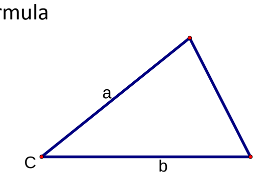
$$B \approx 180 - 51 - 81.17 = \boxed{47.83^\circ}$$



II. Triangle Area

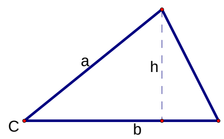
A. Suppose that you know two sides of a triangle and the included angle. The area of the triangle is given by the formula

$$A = \frac{ab}{2}\sin C$$



B. Proof:

We draw a perpendicular segment from the vertex opposite side *a*. We have:



$$\sin C = \frac{h}{a}$$

$$a \sin C = h$$

$$Area = \frac{1}{2}bh = \frac{1}{2}ab\sin C$$

QED

C. Example 2: Find the area of a triangle with sides of length 22 feet and 31 feet, with an included angle of 37 degrees.

$$A = \frac{ab}{2}\sin C$$

$$A = \frac{(22)(31)}{2}\sin 37$$

$$\boxed{A \approx 205.22\, ft^2}$$

D. Heron's Formula: Suppose that a triangle has side lengths *a*, *b*, and *c*. Then the area of the triangle is given by the formula below.

$$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$$

where $s = \frac{1}{2}(a+b+c)$

E. Example 3: Suppose that a triangle has sides of length 12 cm, 15 cm, and 11 cm. Use Heron's formula to estimate the area of the triangle.

$$s = \frac{1}{2}(12+15+11) \;=19$$

$$Area = \sqrt{19(19-12)(19-15)(19-11)}$$

$$Area = \sqrt{19(7)(4)(8)}$$

$$Area = \sqrt{4256}$$

$$\boxed{Area \approx 65.42cm^2}$$