E & M Free-Response Explanations

1. (a) Use Gauss's law with spherical symmetry to find $\vec{E}$ in the three regions. In each case, choose a sphere concentric with the nonconducting shell as the Gaussian surface.

$$\oint_{area} \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$E(4\pi r^2) = \frac{q_{in}}{\varepsilon_0}$$

(i) $r > 2a,\ q_{in} = q_{\text{total}} = \rho \frac{4}{3}\pi((2a)^3 - a^3) = \rho \frac{4}{3}\pi(7a^3)$

$$E_i = \frac{q_{\text{total}}}{4\pi\varepsilon_0 r^2} = \frac{\rho(7a^3)}{3\varepsilon_0 r^2}$$

(ii) $2a > r > a,\ q_{\text{in}} = \rho \frac{4}{3}\pi(r^3 - a^3)$

$$E_{ii} = \frac{\rho \frac{4}{3}\pi(r^3 - a^3)}{4\pi\varepsilon_0 r^2} = \frac{\rho}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right)$$

(iii) $r < a,\ q_{\text{in}} = 0$

$E_{iii} = 0$

(b) Use the definition of electric potential, integrating the field from infinity to the point in question.

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

(i) For $r > 2a$, the distribution behaves as if the total charge were at the center of symmetry.

$$V_i(r) = -\int_{\infty}^{r} \frac{q_{\text{total}}}{4\pi\varepsilon_0 r^2} dr = \frac{q_{\text{total}}}{4\pi\varepsilon_0 r} = \frac{\rho(7a^3)}{3\varepsilon_0 r}$$

(ii) $V_{ii}(r) = -\int_{\infty}^{r} \vec{E} \bullet d\vec{l} = -\int_{\infty}^{2a} \vec{E} \bullet d\vec{l} - \int_{2a}^{r} \vec{E}_{ii} \bullet d\vec{l} = V_i(2a) - \int_{2a}^{r} \frac{\rho}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right) dr$

$$V_{ii}(r) = \frac{\rho(7a^3)}{3\varepsilon_0(2a)} - \frac{\rho}{3\varepsilon_0}\left(\frac{r^2}{2} - \frac{(2a)^2}{2} + \frac{a^3}{r} - \frac{a^3}{2a}\right) = \frac{\rho}{3\varepsilon_0}\left(6a^2 - \frac{r^2}{2} - \frac{a^3}{r}\right)$$

(iii) Since the field is 0 inside, the potential will be constant, equal to the value at the inner surface of the nonconductor.

$$V_{iii}(r) = V_{ii}(2a) = \frac{3\rho a^2}{2\varepsilon_0}$$

(c) When the conducting sphere is put in place, the electric field will be the same everywhere except for the region within the conductor itself, where the field will be 0. This follows from Gauss's law and the fact that the conductor carries no excess charge. Integrating the field in from infinity in this case, you get a different contribution in the region only from $d \rightarrow c$, which now gives a 0 contribution. The potential difference at the center will now be *less* by an amount equal to the old potential difference from $d \rightarrow c$.

2. (a) At $t = 0$, the 12 μF capacitor behaves as a short circuit. The left loop has a resistance of 14 $M\Omega$, and

$$i = \frac{20}{14 \times 10^6} = 1.43\ \mu\text{A}$$

(b) When fully charged, the 12 μF capacitor has a 20 V potential difference.

$$Q_{\text{max}} = CV = 12(20) = 240\ \mu\text{C}$$

(c) Use the capacitor energy formula.

$$U = \frac{1}{2}CV^2 = \frac{1}{2}(12)(20)^2 = 2{,}400\ \mu\text{J}$$

(d) Just after $S2$ is closed, the 4 μF capacitor will have 0 potential drop, while the 12μF will have 20 V. Applying the loop law to the right loop gives you

$$20 - 10 \times 10^6 i = 0 \quad \Rightarrow i = 2\ \mu\text{A}$$

(e) More generally, if q resides on the 4 μF, then $Q_{\text{max}} - q$ resides on the 12 μF, and $i = \frac{dq}{dt}$. The loop law then gives you

$$\frac{Q_{\text{max}} - q}{12 \times 10^{-6}} - 10 \times 10^6 i - \frac{q}{4 \times 10^{-6}} = 0 \Rightarrow \frac{Q_{\text{max}} - q}{12} - 10i - \frac{q}{4} = 0$$

$$\frac{dq}{dt} + \frac{q}{30} = 2 \qquad q \text{ in } \mu\text{C units}$$

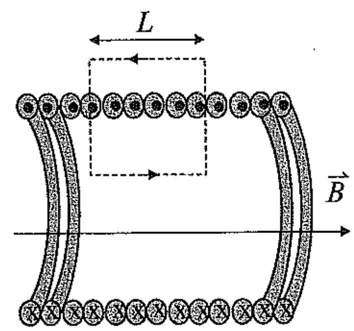
(f) This is a charging capacitor equation with $q_{max} = 60\ \mu C$, so the solution is

$$q(t) = 60\left(1 - e^{-\frac{t}{30}}\right)$$

$$i(t) = \frac{dq}{dt} = 2e^{-\frac{t}{30}} \quad \Rightarrow \quad i(t = 60) = 2e^{-2} = 0.271\ \mu A$$

(g) Since the 4 μF ends with $q_{max} = 60\ \mu C$, the 12 μF must end with 180 μC to conserve charge. Each capacitor will have a potential drop of 15 V, so the final energy stored is

$$U = \frac{1}{2}(12)(15)^2 + \frac{1}{2}(4)(15)^2 = 1{,}800\ \mu J$$



3. (a) Applying Ampere's law to the path shown in dotted lines and recognizing that the field is 0 outside the solenoid, you have

$$\oint_{\text{path}} \vec{B} \cdot d\vec{L} = \mu_0 i_{en}$$

$$BL = \mu_0 nLi \quad \Rightarrow \quad B = \mu_0 ni$$

(b) The flux through the smaller loop will be

$$\Phi = B\pi b^2 = \mu_0 ni\pi b^2 = \mu_0 n\pi b^2 i_0 \cos \omega t$$

Then from Faraday's law, you have

$$V_{in} = -\frac{d\Phi}{dt} = \mu_0 n\pi b^2 \omega i_0 \sin \omega t$$

Then, the induced current is

$$i_b = \frac{V_{in}}{R_b} = \frac{\mu_0 n \pi b^2 \omega i_0}{R_b} \sin \omega t$$

(c) To find the induced electric field, use Faraday's law for the fields.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d}{dt}\oint \vec{B} \cdot \vec{dA}$$

$$E(2\pi b) = \mu_0 n \pi b^2 \omega i_0 \sin \omega t$$

$$E = \frac{1}{2}\mu_0 n b \omega i_0 \sin \omega t$$

(d) For the loop outside the solenoid, the flux is contained within a radius a. From Faraday's law, you have

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d}{dt}\oint \vec{B} \cdot \vec{dA}$$

$$E(2\pi c) = \mu_0 n \pi a^2 \omega i_0 \sin \omega t$$

$$E = \frac{1}{2c}\mu_0 n a \omega i_0 \sin \omega t$$